Fazni prostor

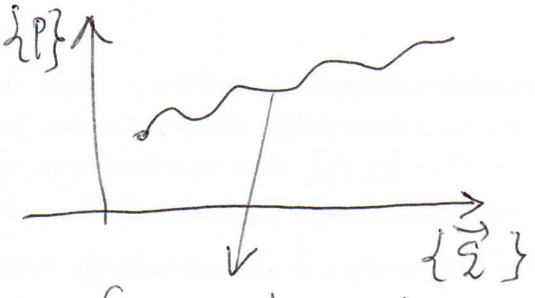
- Apstraktni 6N-dimenzionalni prostor čije su koordinate q_i i p_i i koji ima metriku

$$ds^2 = \sum_{i=1}^{3N} \left(dq_i^2 + dp_i^2\right).$$

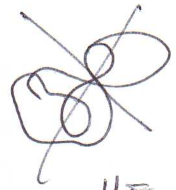
$$(\vec{p}, \vec{q}) \equiv (\vec{p}_1, \vec{p}_2, \dots \vec{p}_N, \vec{q}_1, \vec{q}_2, \dots \vec{q}_N)$$

↓

~~fazna~~ fazna tačka
mikrostanje



fazna trajektorija

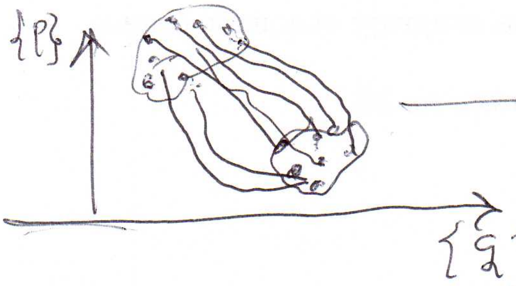


NE
(klasični kauzalitet)

- Koncept ansambla - statistički sistem ima mnogo stepeni slobode

↓

rešava problem početnih uslova

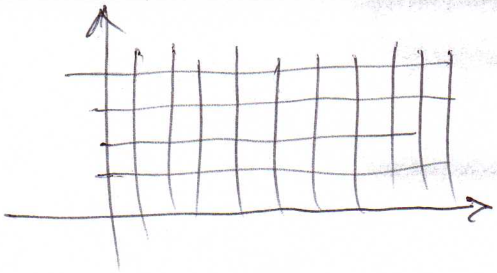


→ nestišljivi fazni fluid

- Klasično gledano sve fazne tačke su međusobno različive, ali kvantna mehanika uvodi ograničenja na različivost

$\Delta \vec{p}\, \Delta \vec{q} \geq h^{3N}$ ← relacija neodređenosti

→ elementarna fazna ćelija



→ fazni prostor izdeljen na fazne ćelije

$N!$ ← identičnost kvantnih čestica

U delu faznog prostora $d\vec{p}\,d\vec{q}$ ima

$$d\Gamma = \frac{d\vec{p}\,d\vec{q}}{N!\,h^{3N}}$$

različitih mikrostanja

Γ – fazna zapremina koja odgovara datom statističkom ansamblu (sva moguća mikrostanja

– Fazna gustina verovatnoće

$$f(\vec{p},\vec{q},t) = \frac{dw}{d\Gamma}$$

(bezdimenziona veličina)

– Fazni fluid zadovoljava Liovill-ovu j-nu (*)

$$\frac{\partial f}{\partial t} + \nabla\cdot(f\vec{v}) = 0$$

— sačuvanje verovatnoće

$$\vec{v} = (\dot{\vec{q}}_1, \ldots \dot{\vec{q}}_N, \dot{\vec{p}}_1, \dot{\vec{p}}_2, \ldots \dot{\vec{p}}_N)$$

– vektor brzine fazne tačke u faznom prostoru

$$\frac{\partial f}{\partial t} + [f,H]_{PZ} = 0 \qquad (*)$$

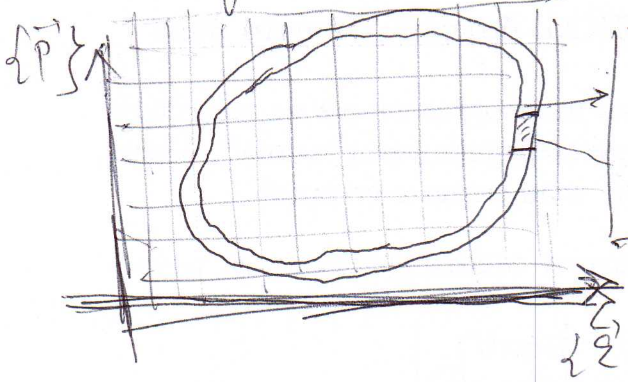
– Tipovi ansambla:

1. Mikrokanonski,
2. Kanonski,
3. Veliki kanonski.

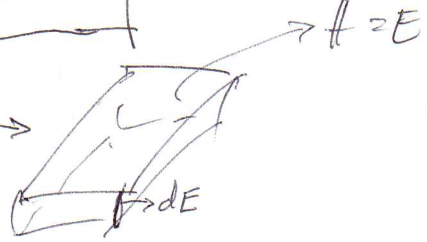
– U ovom kursu od interesa su ravnotežni ansambli $\frac{\partial f}{\partial t} = 0 \Rightarrow f = f(\vec{p},\vec{q})$

$[f, H]_{PZ} = 0$ – fazna gustina verovatnoće kao integral kretanja

– Fazna zapremina i energijska hiperpovrš



$$\Gamma = \int_{E \le H \le E+dE} d\Gamma$$



$$\Gamma(E+\Delta E) = \Gamma(E) + \frac{\partial \Gamma(E)}{\partial E}\Delta E \; ; \; dE \ll E$$

$$\Gamma(E+\Delta E) - \Gamma(E) = \frac{\partial \Gamma(E)}{\partial E}\Delta E$$

$$\boxed{\Omega(E) = \frac{\partial \Gamma(E)}{\partial E}}$$ površina fazne zapremine

---

Dodatni komentari

– Važenje Lagranžove j-ne je ekvivalentno važenju Hamiltonovih j-na

$$\dot{\vec{p}}_i = -\frac{\partial H}{\partial \vec{q}_i}, \quad \dot{\vec{q}}_i = \frac{\partial H}{\partial \vec{p}_i}, \quad i = \overline{1,N},$$

pri čemu potencijali mogu biti, u najopštijem slučaju f-je koordinata, brzina i vremena (misli se na fazni prostor):

$$\vec{Q}_i = -\frac{\partial U}{\partial \vec{q}_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec{q}}_i} \; ;$$

konzervativne sile $U(\{\vec q\})$

potencijalne sile $U(\{\vec q\}, t)$

generalisano potencijalne sile $U(\{\dot{\vec q}\}, \{\vec q\}$

- Statistička mehanika počiva na važenju ERGODIČNE hipoteze: Srednje vrednosti po vremenu jednake su srednjim vrednostima po ansamblu, odnosno

$$\langle A \rangle_t = \langle A \rangle_\Gamma$$

$$\langle A \rangle_t = \frac{1}{t} \int_0^t A(\tau)\, d\tau$$

$$\langle A \rangle_\Gamma = \int_\Gamma A(\vec{p}, \vec{q})\, f(\vec{p}, \vec{q})\, d\Gamma$$

(podrazumeva se ravnotežna statistička mehanika)

- Ansambl ≡ Statistički ansambl ≡ Gibbs-ov ansambl

- $d\Gamma = d\vec{p}\, d\vec{q}$ VS $d\Gamma = \dfrac{d\vec{p}\, d\vec{q}}{N!\, h^{3N}}$

ZZ

$d\Gamma$ Element fazne zapremine

1 Generalisati Liouvill-ovu teoremu za slučaj kada na posmatrani sistem deluju i nepotencijalne sile i formulisati uslov koji bi morale zadovoljavati ove nepotencijalne sile da bi fazna gustina verovatnoće opadala sa vremenom.

$$\dot{\vec{p_i}} = -\frac{\partial H}{\partial \vec{q_i}} + Q_i^* \quad , \quad \dot{\vec{q_i}} = \frac{\partial H}{\partial \vec{p_i}}$$

$$\frac{\partial f}{\partial t} + \nabla (f \vec{v}) = 0$$

$$v = (\dot{\vec{p}}_1, \dots \dot{\vec{p}}_N, \dot{\vec{q}}_1, \dots \dot{\vec{q}}_N)$$

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial (f \dot{\vec{p_i}})}{\partial \vec{p_i}} + \frac{\partial (f \dot{\vec{q_i}})}{\partial \vec{q_i}} \right) = 0$$

$$\underbrace{\frac{\partial f}{\partial t} + \sum_{i=1}^{N} \left(\frac{\partial f}{\partial \vec{p_i}} \dot{\vec{p_i}} + \frac{\partial f}{\partial \vec{q_i}} \dot{\vec{q_i}} \right)}_{\text{totalni izvod po vremenu na kanuc } f \text{ je } f(\vec{q_i}, \vec{p_i}, t)} + f \sum_{i=1}^{N} \left(\frac{\partial \dot{\vec{p_i}}}{\partial \vec{p_i}} + \frac{\partial \dot{\vec{q_i}}}{\partial \vec{q_i}} \right) = 0$$

$$\frac{df}{dt} + f \sum_{i=1}^{N} \left(-\cancel{\frac{\partial^2 H}{\partial \vec{p_i} \partial \vec{q_i}}} + \frac{\partial Q_i^*}{\partial \vec{p_i}} + \cancel{\frac{\partial^2 H}{\partial \vec{q_i} \partial \vec{p_i}}} \right) = 0$$

$$\frac{df}{dt} = -f \sum_{i=1}^{N} \frac{\partial Q_i^*}{\partial \vec{p_i}} \quad ; \quad f(\vec{p_i}, \vec{q}) \geq 0$$

$$\frac{df}{dt} < 0 \Rightarrow \sum_{i=1}^{N} \frac{\partial Q_i^*}{\partial \vec{p_i}} > 0$$

2 Pokazati invarijantnost fazne zapremine za sledeć fizičke sisteme:

a) Slobodna čestica (1D)

b) Harmonijski oscilator (1D) → Позвати се на КМ, где се на формално исти начин изводе $q(t)$ и $p(t)$

---

a) $$\mathcal{H} = \frac{p^2}{2m} \qquad (1D)$$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = 0 \Rightarrow p = \text{const}$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m} \Rightarrow q = \frac{p}{m} t + \text{const}$$

$$p' = p$$

$$q' = \frac{p}{m} t' + \text{const}$$

$$\left.\begin{aligned} q &= \frac{p}{m} t + \text{const} \\ q' &= \frac{p}{m} t' + \text{const} \end{aligned}\right\} \Rightarrow q' - q = \frac{p}{m}(t' - t)$$

$$q' = q + \frac{p}{m}(t' - t)$$

tj.

$p' = p$ i $q' = q + \frac{p}{m}(t' - t)$

$$J(t', t) = \frac{\partial(p', q')}{\partial(p, q)} = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial q'}{\partial q} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & \frac{t' - t}{m} \\ 0 & 1 \end{vmatrix} = 1 \Rightarrow dp' dq' = dp\, dq$$

b) $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2$

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} = -m\omega^2 q$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m}$$, odnosno

$$\dot{p}(t) = -m\omega^2 q(t) \qquad (1)$$

$$\dot{q}(t) = \frac{p(t)}{m} \qquad (2)$$

$(2)/\frac{d}{dt}$ i (1) u (2) $\Rightarrow$

$$\ddot{q}(t) = \frac{\dot{p}(t)}{m} = -\omega^2 q(t) \Rightarrow$$

$$\ddot{q}(t) + \omega^2 q(t) = 0$$

$$k^2 + \omega^2 = 0 \Rightarrow \quad k = \pm i\omega$$

$$q(t) = c_1 e^{i\omega t} + c_2 e^{-i\omega t}$$ + Eulerov obrazac

$$q(t) = A\cos\omega t + B\sin\omega t \qquad (3)$$

$$t = 0 \quad \Rightarrow q(0) = A$$

$(3) \Rightarrow \quad \dot{q}(t) = -A\omega\sin\omega t + B\omega\cos\omega t$

Zbog (2)

$$\dot{q}(0) = \frac{p(0)}{m} = B\omega \Rightarrow \quad B = \frac{p(0)}{m\omega}$$

Onda je (3) konačno

$$\boxed{q(t) = q(0)\cos\omega t + \frac{p(0)}{m\omega}\sin\omega t} \qquad (4)$$

$(1)/\frac{d}{dt}$ i (2) u (1) $\Rightarrow$

$$\ddot{p}(t) = -m\omega^2 \dot{q}(t) = -m\omega^2 \frac{\dot{p}(t)}{m} = -\omega^2 p(t)$$

$$\ddot{p}(t) + \omega^2 p(t) = 0$$

$$p(t) = C\cos\omega t + D\sin\omega t \qquad (5)$$

$$p(0) = C$$

$(5) \Rightarrow$ $\dot{p}(t) = -C\omega\sin\omega t + D\omega\cos\omega t$

Zbog (1)

$$-m\omega^2 q(0) = D\omega \Rightarrow D = -m\omega\, q(0)$$

Onda je (5)

$$\boxed{p(t) = p(0)\cos\omega t - m\omega\, q(0)\sin\omega t} \qquad (6)$$

$$p' = p\cos\omega t - m\omega q\sin\omega t$$

$$q' = q\cos\omega t + \frac{p}{m\omega}\sin\omega t$$

$$J(t', t) = \frac{\partial(p', q')}{\partial(p, q)} = \begin{vmatrix} \frac{\partial p'}{\partial p} & \frac{\partial q'}{\partial p} \\ \frac{\partial p'}{\partial q} & \frac{\partial q'}{\partial q} \end{vmatrix} =$$

$$= \begin{vmatrix} \cos\omega t & -m\omega\sin\omega t \\ \frac{1}{m\omega}\sin\omega t & \cos\omega t \end{vmatrix} = \cos^2\omega t + \sin^2\omega t = 1$$

pa je $\boxed{dp'dq' = dpdq}$

3. U delu faznog prostora koji sadrži zapreminu $\Gamma(E)$ ograničenu hiperpovršinom konstantne energije datog fizičkog sistema je $f(q_1 \ldots q_s, p_1 \ldots p_s)$ jedna poznata integrabilna f-ja. Dokazati ispravnost sledeće relacije

$$\frac{d}{dE} \int_{\Gamma(E)} f(q_1 \ldots p_s)\, d\Gamma = \int_{\Sigma(E)} f(q_1, \ldots p_s) \frac{d\Sigma}{|\mathrm{grad} H|}$$

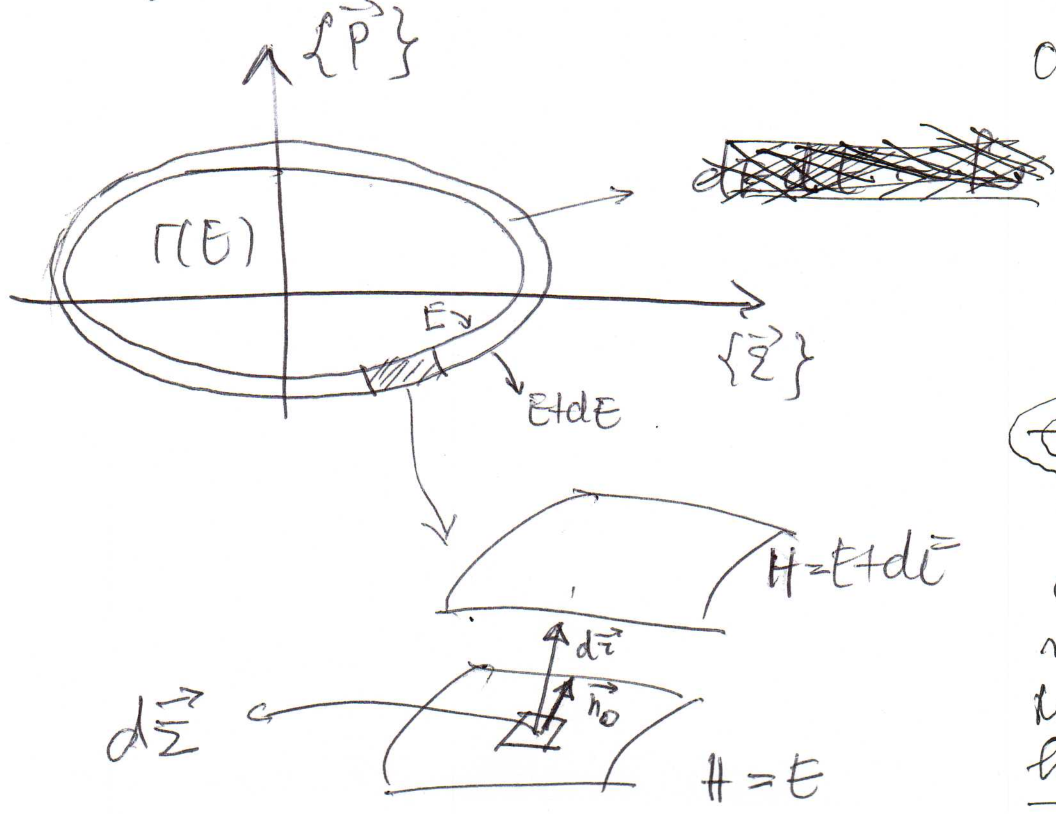
zapreminski integral

površinski integral

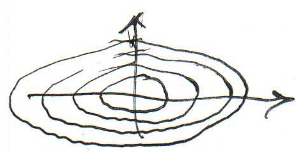
Ovde je $|\mathrm{grad} H|$ neposredna generalizacija analogne veličine iz Euklidovog prostora, tj.

$$|\mathrm{grad} H| = \left[\sum_{i=1}^{s} \left(\frac{\partial H}{\partial q_i} \right)^2 + \sum_{i=1}^{s} \left(\frac{\partial H}{\partial p_i} \right)^2 \right]^{\frac{1}{2}}$$

Ovde je $d\Gamma = d\vec{p}_s\, d\vec{q}_s$!



Primedba: Ova slika liči na onu koja je od interesa za MKA ali nije identična



Ovde je od interesa cela fazna zapremina, a kod MKA je od interesa hiperpovrš

$$d\Gamma = d\vec{\Sigma}\cdot d\vec{r}$$

$$= d\Sigma\,\underline{\vec{n}_0\cdot d\vec{r}}$$

$$d\mathcal{H} = dE = \operatorname{grad}\mathcal{H}\cdot d\vec{r}$$

$$= |\operatorname{grad}\mathcal{H}|\,\vec{n}_0\cdot d\vec{r}$$

$$\underline{\vec{n}_0\cdot d\vec{r}} = \frac{dE}{|\operatorname{grad}\mathcal{H}|}$$

$$d\Gamma = \frac{d\Sigma\, dE}{|\operatorname{grad}\mathcal{H}|}$$

$$\int_{\Gamma(E)} f(\vec{p},\vec{\varepsilon})\,d\Gamma = \int_{\Gamma(E)} f(\vec{p},\vec{\varepsilon})\,\frac{d\Sigma\, dE}{|\operatorname{grad}\mathcal{H}|}$$

apariti da je $\int_{E_{min}}^{E} dE = E - E_{min}$
a je zato diferenciranje
, E lako!

$$= \int_{E_{min}}^{E} dE \int_{\Sigma(E)} \frac{f(\vec{p},\vec{\varepsilon})\,d\Sigma}{|\operatorname{grad}\mathcal{H}|}$$

$$\frac{d}{dE}\int_{\Gamma(E)} f(\vec{p},\vec{\varepsilon})\,d\Gamma = \int_{\Sigma(E)} \cancel{f(\vec{p},\vec{\varepsilon})}\,\frac{d\Sigma}{|\operatorname{grad}\mathcal{H}|}$$

Q.E.D.

4. Izračunati zapreminu i površinu n-dimenzionalne hipersfere u Euklidskom prostoru

Zapremina

$d=2 \qquad V_2 = R^2\pi$

$d=3 \qquad V_3 = \frac{4}{3} R^3 \pi$

} analogija

$$V_n = c_n R^n \longrightarrow dV_n = n c_n R^{n-1} dR$$

$$V_n = \underset{x_1^2 + \dots x_n^2 = R^2}{\int \dots \int} dx_1 \dots dx_n$$

$$c_n = ?$$

Podjimo od Poisson-ovog integrala

$$I = \int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$$

Množeći n ovakvih integrala za svako x_i, dobijamo

$$\pi^{\frac{n}{2}} = \int_{x_i=-\infty}^{+\infty} \dots \int e^{-\left(\sum_{i=1}^{n} x_i^2\right)} \underbrace{\prod_{i=1}^{n} dx_i}_{dV_n}$$

$$= \int_0^{\infty} e^{-R^2} n c_n R^{n-1} dR$$

$$= n c_n \int_0^{\infty} e^{-R^2} R^{n-1} dR$$

Smena:

$$R^2 = t \quad , \quad R = \sqrt{t} \longrightarrow dR = \frac{dt}{2\sqrt{t}}$$

$$\pi^{\frac{n}{2}} = \frac{nc_n}{2} \int_0^\infty e^{-t} \, t^{\frac{n}{2}-1} dt$$

$$= \frac{nC_n}{2} \Gamma\left(\frac{n}{2}\right)$$

$$C_n = \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)} = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

Konačno,

$$\boxed{V_n = C_n R^n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} R^n}$$

Domaći:

Npr, za $n=3$

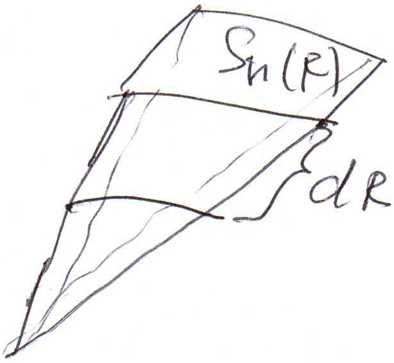
$$V_3 = \frac{\pi^{\frac{3}{2}} R^3}{\Gamma\left(\frac{3}{2}+1\right)} = \frac{\pi^{\frac{3}{2}} R^3}{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)} = \frac{\pi^{\frac{3}{2}} R^3}{\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)}$$

$$= \frac{\pi^{\frac{3}{2}} R^3}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{\pi^{3/2} R^3}{\frac{3}{4}\pi^{1/2}} = \frac{4}{3} R^3 \pi$$

Površina

$$S_n(R) = nC_n R^{n-1}$$

Od ranije $dV_n = nC_n R^{n-1}\, dR$



$$dV_n = S_n(R)\, dR$$

$$S_n(R) = n \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)} R^{n-1}$$

$$= \frac{n\,\pi^{n/2}}{\frac{n}{2}\,\Gamma(\frac{n}{2})} R^{n-1}$$

$$\boxed{S_n(R) = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})} R^{n-1}}$$

Domaći: Koliko je $S_3(R)$? [$4\pi R^2$]

Domaći: Kolika je zapremina n-dimenzionalnog hiperelipsoida opisanog jednačinom

$$\left(\frac{x_1}{a_1}\right)^R + \dots \left(\frac{x_n}{a_n}\right)^2 = 1 \ ?$$

5.- Izračunati veličinu fazne zapremine $\Gamma(E^*, V)$, omeđenu sa hiperpovrši konstantne energije E, za sistem od N molekula idealnog gasa u zapremini V.

Zapremina V kao spoljašnji parametar

$$\Gamma(E^*, V) = \underbrace{\int \cdots \int}_{3N} dx_1 dy_1 dz_1 \ldots dx_N dy_N dz_N$$

$$\cdots \underbrace{\int \cdots \int}_{3N} dp_{x_1} dp_{y_1} dp_{z_1} \ldots dp_{x_N} dp_{y_N} dp_{z_N}$$

$$H = \sum_{i=1}^{N} \frac{1}{2m} \left(p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2 \right)$$

$$\Gamma(E^*, V) = V^N \underbrace{\int \cdots \int}_{3N} dp_{x_1} dp_{y_1} dp_{z_1} \ldots dp_{x_N} dp_{y_N} dp_{z_N}$$

Integracija po delu zapremine impulsnog potprostora faznog prostora

$$\sum_{i=1}^{N} \left(p_{x_i}^2 + p_{y_i}^2 + p_{z_i}^2 \right) \leq 2mE^*$$

$$R = \sqrt{2mE^*}$$

$$\Gamma(E^*, V) = V^N \frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}+1\right)} R^{3N}$$

odnosno

$$\Gamma(E^*, V) = V^N \frac{\pi^{3N/2}}{\Gamma\left(\frac{3N}{2}+1\right)} \left(2mE^*\right)^{\frac{3N}{2}}$$

Domaći: Kolika je površina ove fazne zapremine?

$$\Omega(E^*) = \left.\frac{\partial \Gamma(E)}{\partial E}\right|_{E=E^*}$$

6 Izračunati veličinu fazne zapremine $\Gamma(E)$ omeđenu sa hiperpovrši konstantne energije E za sistem od N neinteragujućih LHO iste frekvence ω.

J-na hiperpovrši konstantne energije za ovaj sistem

$$\sum_{i=1}^{N}\left(\frac{1}{2m}p_i^2+\frac{m\omega^2}{2}q_i^2\right)=E$$

$$\Downarrow$$

$$\sum_{i=1}^{N} p_i^2+(m\omega q_i)^2=2mE$$

$$\sum_{i=1}^{N} p_i^2+q_i^{*2}=2mE$$

$$q_i^*=m\omega q_i \Rightarrow dq_i=\frac{dq_i^*}{m\omega},\ \forall i$$

$$\Gamma(E)=\underbrace{\int\dots\int}_{2N} dq_1\dots dq_N\, dp_1\dots dp_N$$

$$\Gamma(E)=\frac{1}{(m\omega)^N}\underbrace{\int\dots\int}_{2N} dq_1^*\,dq_2^*\dots dq_N^*\,dp_1\dots dp_N$$

$R=\sqrt{2mE}$ – poluprečnik hipersfere

$$\Gamma(E)=\frac{1}{(m\omega)^N}\,\frac{R^{2N}\pi^{\frac{2N}{2}}}{\Gamma\left(\frac{2N}{2}+1\right)}$$

$$\Gamma(E) = \frac{1}{(m\omega)^N} \frac{(2mE)^{2N/2} \pi^N}{\Gamma(N+1)}$$

$$\Gamma(E) = \frac{1}{(m\omega)^N} \frac{(2m\pi E)^N}{N!} = \frac{1}{N!}\left(\frac{2\pi E}{\omega}\right)^N$$

Domaći: Kolika je površina ove fazne zapremine?

$$\Omega(E) = \frac{\partial \Gamma(E)}{\partial E}$$

7. Ako je poznato da se element fazne zapremine može izraziti kao

$$d\Gamma = \frac{d\Sigma \, dE}{|\mathrm{grad}\,\mathcal{H}|}$$

izračunati verovatnoću nalaženja klasičnog LHO amplitude A u nekoj tački između q i $q+dq$.

Ovde je od interesa verovatnoća nalaženja fazne tačke na nekom delu energijske hiperpovrši. Znajući da važi

$$\Gamma(E) = \int_{\mathcal{H} \leq E} d\Gamma = \int_{\mathcal{H} \leq E} \frac{d\Sigma \, dE}{|\mathrm{grad}\,\mathcal{H}|}$$

$$= \int_{E_{min}}^{E} \left(\int_{\mathcal{H}=E} \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|} \right) dE$$

i relacija $\Omega(E) = \frac{\partial \Gamma}{\partial E}$, sledi da je

$$\Omega(E) = \int_{\mathcal{H}=E} \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}$$

da važi i

$$d\Omega(E) = \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}$$

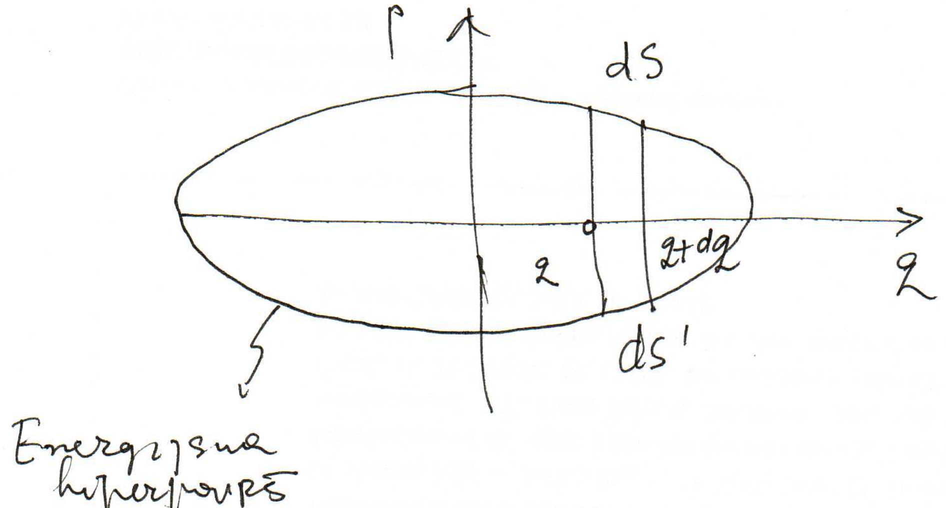
što je odgovarajući diferencijalni element hiperpovrši

Verovatnoća nalaženja fazne tačke na nekom delu hiperpovrši

$$dw = \frac{d\Omega(E)}{\Omega(E)} = \frac{\dfrac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}}{\displaystyle\int_{\mathcal{H}=E} \frac{d\Sigma}{|\mathrm{grad}\,\mathcal{H}|}}$$

Fazna trajektorija LHO-a data je jednačinom

$$\frac{q^2}{\dfrac{2E}{m\omega^2}} + \frac{p^2}{2mE} = 1$$



$$ds \sim d\Sigma$$

$$dw = 2\,\frac{\dfrac{ds}{|\mathrm{grad}\,\mathcal{H}|}}{\displaystyle\int_L \frac{ds}{|\mathrm{grad}\,\mathcal{H}|}}$$

Verovatnoće nalaženja fazne tačke na ds i ds' su jednake

$$ds = \sqrt{dq^2 + dp^2} = \sqrt{1 + \left(\frac{dp}{dq}\right)^2}\, dq$$

$$\frac{p^2}{2m} + \frac{1}{2} m f^2 q^2 = E \quad , \quad p = \sqrt{2mE - m^2 f^2 q^2}$$

$$\frac{dp}{dq} = - \frac{m^2 f^2 q}{\sqrt{2mE - m^2 f^2 q^2}}$$

$$ds = \sqrt{1 + \left(m^2 f^2 \frac{q}{p}\right)^2}\, dq$$

$$|\operatorname{grad} \mathcal{H}| = \sqrt{\left(\frac{\partial \mathcal{H}}{\partial p}\right)^2 + \left(\frac{\partial \mathcal{H}}{\partial q}\right)^2}$$

$$= \frac{p}{m} \sqrt{1 + \left(m^2 f^2 \frac{q}{p}\right)^2}$$

$$dw = 2 \frac{\frac{dq}{\frac{p}{m}}}{\int_L \frac{dq}{\frac{p}{m}}} = 2 \cdot \frac{\frac{dq}{v}}{\int_L \frac{dq}{v}}$$

$$\int_L \frac{dq}{v} = T = \frac{2\pi}{f}$$

Zasada zachowania energii

$$\frac{1}{2} m \dot{q}^2 + \frac{1}{2} m f^2 q^2 = \frac{1}{2} m f^2 A^2 \Rightarrow$$

$$\dot{q} = \sqrt{f^2 (A^2 - q^2)} = v$$

$$dw = 2 \frac{\dfrac{dq}{\sqrt{f^2(A^2-q^2)}}}{\dfrac{2\pi}{f}} = \frac{1}{\pi} \frac{dq}{\sqrt{A^2-q^2}}$$

Uporedi sa odgovarajućim zadatkom iz prethodne glave

8. Izračunati zapreminu u faznom prostoru i odgovarajuću površinu te zapremine za trodimenzionalni ultrarelativistički klasični gas koji se sastoji od N čestica fiksirane energije i nalazi se u zapremini V.

$$E^2 = p^2c^2 + m_0^2c^4$$ ← Energija čestice u relativističkoj mehanici

Za ultrarelativističke čestice energija mirovanja može biti zanemarena pa sledi

$$E = pc$$

Za N čestica Hamiltonijan onda pišemo kao

$$H = c\sum_{i=1}^{N} p_i \qquad p_i = |\vec{p_i}|$$

$$\sum_i p_i \leq \frac{E}{c}$$ – jednačina hiperpovrši

$$\Gamma(E) = \int \dots \int \prod_{i=1}^{N} d\vec{q_i}\, d\vec{p_i}$$

Hamiltonijan ne zavisi od koordinata $\vec{q_i}$ pa važi

$$\int d\vec{q}_i\, d\vec{p}_i = \int d^3 q_i\, d^3 p_i = V \int d^3 p_i$$

$$= 4\pi V \int p_i^2 dp_i \,, \quad \forall i$$

pa je

$$T(E) = (4\pi V)^N \int \cdots \int \prod_{i=1}^{N} p_i^2 dp_i \equiv (4\pi V)^N I_N(E) \qquad (*)$$

Ako impuls jedne čestice fiksiramo da bude p, onda zbog

$$E = c \sum_{i=1}^{N} p_i = cp + c \sum_{i=1}^{N-1} p_i$$

$$E' = c \sum_{i=1}^{N-1} p_i = E - cp \,;$$

Također

$$I_N(E) = \int_0^{E/c} dp\, p^2\, I_{N-1}(E - pc) \qquad (**)$$

Budući da je $(*)$

$$I_N(E) = \int \cdots \int \prod_{i=1}^{N} p_i^2 dp_i \,, \quad \text{slijedi da je}$$

$$I_N(E) = \mathcal{O}\left(\left(\frac{E}{c}\right)^{3N}\right) \quad \text{tj.}$$

→ v. pozadi

$$I_N(E) = C_N \left(\frac{E}{c}\right)^{3N}$$ a odavde

$$I_{N-1}(E) = C_{N-1} \left(\frac{E}{c}\right)^{3N-3}$$ i dalje

$$I_{N-1}(E-pc) = C_{N-1} \left(\frac{E}{c} - p\right)^{3N-3}$$

$\downarrow$

Ovo vraćamo u $(**)$ i sledi

$$I_N(E) = \int_0^{E/c} dp\, p^2\, C_{N-1} \left(\frac{E}{c} - p\right)^{3N-3}$$

tj:

$$C_N \left(\frac{E}{c}\right)^{3N} = \int_0^{E/c} dp\, p^2\, C_{N-1} \left(\frac{E}{c} - p\right)^{3N-3}$$

$\downarrow$

$$C_N = \left(\frac{c}{E}\right)^{3N} C_{N-1} \int_0^{E/c} dp\, p^2 \left(\frac{E}{c} - p\right)^{3N-3}$$

$$= \left(\frac{c}{E}\right)^{3N} C_{N-1} \int_0^{E/c} dp\, p^2 \left(\frac{E}{c}\right)^{3N-3} \left(1 - \frac{pc}{E}\right)^{3N-3}$$

$$= \left(\frac{c}{E}\right)^{3} C_{N-1} \int_0^{E/c} dp\, p^2 \left(1 - \frac{pc}{E}\right)^{3N-3}$$

Smena $\frac{pc}{E} = x \Rightarrow dp = \frac{E}{c} dx$

$\downarrow$

$[0,1]$

$$C_N = \left(\frac{c}{E}\right)^3 C_{N-1} \int_0^1 \frac{E}{c} dx \, \frac{E^2}{c^2} \, x^2 (1-x)^{3N-3}$$

$$= C_{N-1} \int_0^1 dx \, x^2 (1-x)^{3N-3}$$

Beta f-ja

$$B(n,m) = \int_0^1 dx \, x^{n-1} (1-x)^{m-1}$$

$$B(n,m) = \frac{\Gamma(n)\,\Gamma(m)}{\Gamma(n+m)}$$

Gama f-ja

$n = 3 \qquad m = 3N-2$

$$B(3, 3N-2) = \frac{\Gamma(3)\,\Gamma(3N-2)}{\Gamma(3N+1)}$$

$$= 2\,\frac{(3N-3)!}{(3N)!}$$

$$C_N = 2\,\frac{(3N-3)!}{(3N)!}\,C_{N-1}$$

tj.

$$C_N = \frac{2}{3N(3N-1)(3N-2)}\,C_{N-1}$$

Rekurzivno

$$C_{N-1} = \frac{2}{(3N-3)(3N-4)(3N-5)}\,C_{N-2}$$

$$C_{N-2} = \frac{2}{(3N-6)(3N-7)(3N-8)}\,C_{N-3}$$

$$\vdots$$

Onda, npr. važi

$$C_N = \frac{2}{3N(3N-1)(3N-2)}\;\frac{2}{(3N-3)(3N-4)(3N-5)}\;\frac{2}{(3N-6)(3N-7)(3N-8)}\,C_{N-3}$$

$$C_N = \frac{2^N}{(3N)!} \Rightarrow C_0 = 1$$

$$I_N(E) = C_N\left(\frac{E}{c}\right)^{3N} = \frac{2^N}{(3N)!}\left(\frac{E}{c}\right)^{3N}$$

$$\Gamma(E) = (4\pi V)^N I_N(E) = \dots = \frac{1}{(3N)!}\left(\frac{8\pi V E^3}{c^3}\right)^N$$